

On the rate of convergence of Krasnosel'skiĭ-Mann iterations and their connection with sums of Bernoullis

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Abstract

In this paper we establish an estimate for the rate of convergence of the Krasnosel'skiĭ-Mann iteration for computing fixed points of non-expansive maps. Our main result settles the Baillon-Bruck conjecture [3] on the asymptotic regularity of this iteration. The proof proceeds by establishing a connection between these iterates and a stochastic process involving sums of non-homogeneous Bernoulli trials. We also exploit a new Hoeffding-type inequality to majorize the expected value of a convex function of these sums using Poisson distributions.

Keywords: asymptotic regularity, non-expansive maps, fixed point iteration, sums of Bernoullis, Hoeffding-type inequalities

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1 Introduction

Let $T : C \rightarrow C$ be a non-expansive map defined on a convex subset $C \subseteq X$ of a normed space $(X, \|\cdot\|)$. The Krasnosel'skiĭ-Mann iteration for computing a fixed-point of T is defined by (cf. [22, 23])

$$x_k = (1 - \alpha_k)x_{k-1} + \alpha_k T x_{k-1} \quad (1)$$

with $x_0 \in C$ given and $\alpha_k \in [0, 1]$.

Strong convergence of x_k to a fixed point was proved in [22, Krasnosel'skiĭ] for $\alpha_k \equiv \frac{1}{2}$, when X is a uniformly convex Banach space and $T(C)$ is contained in a compact subset of C . This result was extended to $\alpha_k \equiv \alpha$ [28, Schaefer] and X strictly convex [9, Edelstein], while [17, Ishikawa] proved it for general Banach spaces with α_k bounded away from 1 and $\sum \alpha_k = \infty$. The Banach case with $\alpha_k \equiv \alpha$ was also considered in [10, Edelstein and O'Brien]. Without the compactness assumption, weak convergence was established in [25, Reich] assuming $\sum \alpha_k(1 - \alpha_k) = \infty$ and $\text{Fix}(T) \neq \emptyset$, for X uniformly convex with a Fréchet differentiable norm. Although strong convergence does not hold in general (see [12, Genel and Lindenstrauss] and [5, Bauschke *et al.*]), it does occur for most operators in the sense of Baire's categories (see [27, Reich and Zaslavski]).

The crucial step in proving the convergence of the iterates in all these results is to show that $\|x_n - T x_n\|$ tends to 0, a property which is now known as *asymptotic regularity* [4, 6, 8, 26]. Under various assumptions, asymptotic regularity was also proved in [15, Groetsch] and [13, Goebel and Kirk]. The latter noted a certain uniformity in the convergence, namely, for each $\epsilon > 0$ we have $\|x_n - T x_n\| \leq \epsilon$ for all $n \geq n_0$, with n_0 depending on ϵ and C but independent of the initial point x_0 and the map T . More recently, using proof mining techniques, Kohlenbach [20, 21] showed that n_0 could be chosen to depend on C only through its diameter. An explicit metric estimate which readily implies all these results was stated in [3, Baillon and Bruck], namely, they conjectured the existence of a universal constant κ such that

$$\|x_n - T x_n\| \leq \kappa \frac{\text{diam}(C)}{\sqrt{\sum_{i=1}^n \alpha_i(1 - \alpha_i)}} \quad (2)$$

and proved it for the case $\alpha_i \equiv \alpha$ with $\kappa = \frac{1}{\sqrt{\pi}}$.

In this paper we settle this conjecture by proving that the bound holds in general with $\kappa = \frac{1}{\sqrt{\pi}}$ for any sequence α_k and each non-expansive $T : C \rightarrow C$.

Although we do not know whether this is the smallest possible κ , we provide an example which shows that it cannot be improved by more than 17%. We also discuss how the result can be used to analyze the convergence of (1), and how it applies when C is unbounded but $\text{Fix}(T) \neq \emptyset$.

Our proof is based on a recursive bound for the distances between the iterates $\|x_m - x_n\| \leq c_{mn}$, where c_{mn} admits a nice probabilistic interpretation in terms of a random walk on \mathbb{Z} . In proving the theorem we exploit some properties of the hypergeometric and modified Bessel functions, as well as a known identity for Catalan numbers. We also use the following Hoeffding-type inequality which might be of interest on its own: *if $S = X_1 + \dots + X_m$ is a sum of independent Bernoullis and Z is a Poisson with the same mean $\mathbb{E}(Z) = \mathbb{E}(S)$, then $\mathbb{E}[g(S)] \leq \mathbb{E}[g(Z)]$ for every convex function $g : \mathbb{N} \rightarrow \mathbb{R}$.*

2 Main result

Theorem 1. *The Krasnosel'skiĭ-Mann iterates generated by (1) satisfy*

$$\|x_n - Tx_n\| \leq \frac{\text{diam}(C)}{\sqrt{\pi \sum_{i=1}^n \alpha_i (1 - \alpha_i)}}. \quad (3)$$

The proof is split into several intermediate steps. Note that by rescaling the norm, we may assume $\text{diam}(C) = 1$.

2.1 A recursive bound

Let $\rho_k = \prod_{j=1}^k (1 - \alpha_j)$ and $\pi_k^n = \rho_n \frac{\alpha_k}{\rho_k} = \alpha_k \prod_{j=k+1}^n (1 - \alpha_j)$. By convention we also set $\rho_0 = \alpha_0 = 1$, while the term Tx_{-1} is interpreted as x_0 .

Proposition 2. *For $n \geq 0$ we have $x_n = \sum_{k=0}^n \pi_k^n Tx_{k-1}$ and*

$$x_m - x_n = \sum_{j=0}^m \sum_{k=m+1}^n \pi_j^m \pi_k^n [Tx_{j-1} - Tx_{k-1}] \quad \text{for } 0 \leq m \leq n. \quad (4)$$

Proof. Dividing (1) by ρ_k we have $\frac{x_k}{\rho_k} = \frac{x_{k-1}}{\rho_{k-1}} + \frac{\alpha_k}{\rho_k} Tx_{k-1}$ which, when iterated, yields $\frac{x_n}{\rho_n} = x_0 + \sum_{k=1}^n \frac{\alpha_k}{\rho_k} Tx_{k-1}$. Using the conventions $\rho_0 = \alpha_0 = 1$ and $x_0 = Tx_{-1}$ we get precisely $x_n = \sum_{k=0}^n \pi_k^n Tx_{k-1}$. This equality, combined with the identities $\sum_{j=0}^m \pi_j^m = 1$ and $\pi_k^m - \pi_k^n = \sum_{j=m+1}^n \pi_j^n \pi_k^m$, yields

$$\begin{aligned}
x_m - x_n &= \sum_{k=0}^m (\pi_k^m - \pi_k^n) T x_{k-1} - \sum_{k=m+1}^n \pi_k^n T x_{k-1} \\
&= \sum_{k=0}^m \sum_{j=m+1}^n \pi_j^n \pi_k^m T x_{k-1} - \sum_{j=0}^m \sum_{k=m+1}^n \pi_j^m \pi_k^n T x_{k-1}
\end{aligned}$$

so that exchanging j and k in the first double sum we obtain (4). \square

Corollary 3. Define c_{mn} recursively by setting $c_{-1,n} = 1$ for all $n \geq 0$ and

$$(R) \quad c_{mn} = \sum_{j=0}^m \sum_{k=m+1}^n \pi_j^m \pi_k^n c_{j-1,k-1} \quad \text{for } 0 \leq m \leq n.$$

Then $\|x_m - x_n\| \leq c_{mn}$ for all $0 \leq m \leq n$.

Proof. The proof is by induction on n . Suppose that $\|x_j - x_k\| \leq c_{jk}$ holds for all $0 \leq j \leq k \leq n-1$. Using the triangle inequality in (4) we get

$$\|x_m - x_n\| \leq \sum_{j=0}^m \sum_{k=m+1}^n \pi_j^m \pi_k^n \|T x_{j-1} - T x_{k-1}\|. \quad (5)$$

The induction hypothesis gives $\|T x_{j-1} - T x_{k-1}\| \leq \|x_{j-1} - x_{k-1}\| \leq c_{j-1,k-1}$ for $1 \leq j < k$, while for $j = 0$ we have $\|T x_{-1} - T x_{k-1}\| = \|x_0 - T x_{k-1}\| \leq \text{diam}(C) = 1 = c_{-1,k-1}$. Plugging these bounds into (5) and using (R) we deduce $\|x_m - x_n\| \leq c_{mn}$ completing the induction step. \square

Note that for $m = n$ we have $c_{nn} = 0$ and the inequality $\|x_n - x_n\| \leq c_{nn}$ holds trivially. More interestingly, since $\|x_n - x_{n+1}\| = \alpha_{n+1} \|x_n - T x_n\|$ we have $\|x_n - T x_n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} \triangleq P^n$ so that Theorem 1 will follow by showing

$$\sqrt{\sum_{i=1}^n \alpha_i (1 - \alpha_i)} P^n \leq \frac{1}{\sqrt{\pi}}. \quad (6)$$

Our analysis proves that this bound is sharp, so that $\frac{1}{\sqrt{\pi}}$ is the best constant one can get from Corollary 3. This does not exclude the possibility that other techniques might lead to sharper bounds in Theorem 1 (*cf.* [2, Baillon and Bruck]).

2.2 Fox-and-Hare race and a random walk

The recurrence (R) has a probabilistic interpretation. Consider a fox at position n trying to catch a hare located at $m < n$. At each integer $i \in \mathbb{N}$

the fox must jump over a hurdle to reach $i-1$. The jump succeeds with probability $(1-\alpha_i)$ in which case the process repeats, otherwise the fox falls at $i-1$ where it rests to recover from injuries. Thus, starting from n the probability of landing at $k-1$ is precisely π_k^n . The fox catches the hare if it jumps successfully down to m or below. Otherwise, the hare runs toward the burrow located at -1 by following the same rules. The process alternates until either the fox catches the hare, or the hare reaches the burrow.

The recurrence (R) satisfied by c_{mn} characterizes precisely the probability for the hare to reach the burrow safely when the process starts at (m, n) . This is also consistent with the boundary cases $c_{-1,n} = 1$ and $c_{nn} = 0$. Note that $\alpha_0 = 1$ so at $i = 0$ both the fox and hare fall with certainty, landing at -1 . From this interpretation we get the following expression for c_{mn} .

Proposition 4. *Let $(F_i)_{i \in \mathbb{N}}$ and $(H_i)_{i \in \mathbb{N}}$ denote independent Bernoulli trials representing respectively the events that the fox and hare fail at the i -th hurdle, so that $\mathbb{P}(F_i = 1) = \mathbb{P}(H_i = 1) = \alpha_i$. Then*

$$c_{mn} = \mathbb{P}(\sum_{i=k}^n F_i > \sum_{i=k}^m H_i \text{ for all } k = m+1, \dots, 1). \quad (7)$$

In particular, denoting $Z_i = F_i - H_i$ we have

$$P^n = \frac{c_{n,n+1}}{\alpha_{n+1}} = \mathbb{P}(\sum_{i=k}^n Z_i \geq 0 \text{ for } k = n, \dots, 1). \quad (8)$$

Proof. Formula (7) is just a restatement of the fact that the hare wins iff the number of times the fox falls in any interval $\{k, \dots, n\}$ is strictly larger than the number of falls of the hare in $\{k, \dots, m\}$. The expression for P^n follows by noting that the event corresponding to $c_{n,n+1}$ in (7) requires $F_{n+1} = 1$ (take $k = n+1$). \square

Formula (8) has an alternative interpretation. Let $p_i = 2\alpha_i(1-\alpha_i)$ so that Z_i takes values in $\{-1, 0, 1\}$ with probabilities $\frac{p_i}{2}, 1-p_i, \frac{p_i}{2}$. The sums $\sum_{i=k}^n Z_i$ taken in reverse order $k = n, \dots, 1$ define a random walk on \mathbb{Z} where at each stage the process stays at the current position with some probability, and otherwise moves left or right with equal probability as in a standard random walk. Hence, P^n is the probability that the walk remains non-negative over n stages. Conditioning on the total number of stages at which the process effectively moves, this is also the probability that a standard random walk stays non-negative over a random number of stages. Using this interpretation we get the following more explicit formula.

Proposition 5. *Let $M = M_1 + \dots + M_n$ be a sum of independent Bernoullis with success probabilities $\mathbb{P}(M_i = 1) = p_i = 2\alpha_i(1 - \alpha_i)$ and consider the integer function $F(m) = \binom{m}{\lfloor m/2 \rfloor} 2^{-m}$. Then $P^n = \mathbb{E}[F(M)]$.*

Proof. The variable M_i can be interpreted as *move/stay* and Z_i can be expressed as $Z_i = M_i D_i$ with D_i independent variables representing the direction of the movement: $\mathbb{P}(D_i = -1) = \mathbb{P}(D_i = 1) = \frac{1}{2}$. Conditioning on the sum M and using the exchangeability of the variables D_i we obtain

$$\begin{aligned} P^n &= \sum_{m=0}^n \mathbb{P}(\sum_{i=k}^n M_i D_i \geq 0 \text{ for } k = n, \dots, 1 | M = m) \mathbb{P}(M = m) \\ &= \sum_{m=0}^n \mathbb{P}(\sum_{j=1}^m D_j \geq 0 \text{ for } \ell = 1, \dots, m) \mathbb{P}(M = m). \end{aligned}$$

The expression $\mathbb{P}(\sum_{j=1}^m D_j \geq 0 \text{ for } \ell = 1, \dots, m)$ is the probability that a standard random walk started from 0 remains non-negative over m stages. Its value is precisely $F(m)$ [11, Ch. III.3] so the conclusion follows. \square

The next result establishes an alternative recursion satisfied by c_{mn} . This is not used in our proof, but we state in case someone could use it to find a simpler proof of Theorem 1.

Proposition 6. *Denoting $\bar{\alpha}_k = 1 - \alpha_k$, we have the recurrence*

$$c_{mn} = \bar{\alpha}_m c_{m-1,n} + \bar{\alpha}_n c_{m,n-1} + (\alpha_n \alpha_m - \bar{\alpha}_n \bar{\alpha}_m) c_{m-1,n-1}. \quad (9)$$

Proof. Denote $w_{jk} = \pi_j^m \pi_k^n c_{j-1,k-1}$ and let $S = A + B - C - D$ with

$$\begin{aligned} A &= c_{mn} &= \sum_{j=0}^m \sum_{k=m+1}^n w_{jk} \\ B &= \bar{\alpha}_m \bar{\alpha}_n c_{m-1,n-1} &= \sum_{j=0}^{m-1} \sum_{k=m}^{n-1} w_{jk} \\ C &= \bar{\alpha}_m c_{m-1,n} &= \sum_{j=0}^{m-1} \sum_{k=m}^n w_{jk} \\ D &= \bar{\alpha}_n c_{m,n-1} &= \sum_{j=0}^m \sum_{k=m+1}^{n-1} w_{jk}. \end{aligned}$$

Canceling out the common terms we get $S = w_{mn} = \alpha_m \alpha_n c_{m-1,n-1}$ which is exactly (9). \square

2.3 A sharp upper bound

From Proposition 5, the bound (6) is equivalent to showing that

$$R^n(p) \triangleq \sqrt{p_1 + \dots + p_n} \mathbb{E}[F(M_1 + \dots + M_n)] \leq \sqrt{\frac{2}{\pi}}$$

for all n and $0 \leq p_i \leq \frac{1}{2}$. The function $R^n(p)$ is strictly concave in each variable p_i separately, so the maximum is attained at the extreme values $0, \frac{1}{2}$ or at a unique point in $(0, \frac{1}{2})$. Interestingly, all non-extreme coordinates may be taken equal.

Lemma 7. $R^n(p)$ is maximal when $p_i \in \{0, u, \frac{1}{2}\}$ for some $0 < u < \frac{1}{2}$.

Proof. Let p maximize $R^n(p)$ and suppose $p_j = x$ and $p_k = y$ with $x, y \in (0, \frac{1}{2})$ and $x \neq y$. Let $h(k) = \mathbb{E}[F(k + S)]$ where $S = \sum_{i \neq j, k} M_i$ so that

$$\begin{aligned} P^n &= (1-x)(1-y)h(0) + [x(1-y) + y(1-x)]h(1) + xyh(2) \\ &= a + b(x+y) + cxy \end{aligned}$$

with $a = h(0)$, $b = h(1) - h(0)$ and $c = h(0) + h(2) - 2h(1)$. Setting $m = \sum_{i \neq j, k} p_i$ it follows that $x, y \in (0, \frac{1}{2})$ maximize the expression

$$\sqrt{m + x + y} [a + b(x + y) + cxy].$$

Setting the partial derivatives to 0 we get $cx = cy$ and since $x \neq y$ it follows that $c = 0$. But then, the function depends only on the sum $x + y$ and we may change these coordinates to $x + \epsilon, y - \epsilon$ keeping the same value, until one of them hits an extreme value: either $x + \epsilon = \frac{1}{2}$ or $y - \epsilon = 0$. This yields a new optimal p with one coordinate less in $(0, \frac{1}{2})$. Repeating this process we get an optimal p whose coordinates take at most one value in $(0, \frac{1}{2})$. \square

According to this Lemma, in order to bound $R^n(p)$ it suffices to consider the case $p_i \in \{0, u, \frac{1}{2}\}$ with $0 < u < \frac{1}{2}$. Moreover, by changing n we may ignore the deterministic variables with $p_i = 0$. We distinguish two cases.

2.3.1 All coordinates $p_i = u$

In this case $R^n(p) = \sqrt{nu} \mathbb{E}[F(S)]$ with $S \sim B(n, u)$ Binomial. This case follows from the results in [3, Baillon and Bruck] which were obtained using a computer generated proof. Here we provide a direct proof based on a known identity for Catalan numbers.

Proposition 8. Let $S \sim B(n, u)$ with $0 < u < \frac{1}{2}$. Then

$$\mathbb{E}[F(S)] = \sum_{k=0}^n \frac{(-1)^k}{k+1} \binom{2k}{k} \binom{n}{k} \left(\frac{u}{2}\right)^k \quad (10)$$

and $R^n(p) = \sqrt{nu} \mathbb{E}[F(S)]$ increases with n towards $\sqrt{\frac{2}{\pi}}$.

Proof. Using the Binomial theorem, a straightforward computation gives

$$\begin{aligned}
\mathbb{E}[F(S)] &= \sum_{j=0}^n F(j) \binom{n}{j} u^j (1-u)^{n-j} \\
&= \sum_{j=0}^n F(j) \binom{n}{j} u^j \sum_{i=0}^{n-j} \binom{n-j}{i} (-u)^i \\
&= \sum_{j=0}^n \sum_{k=j}^n (-1)^j F(j) \binom{n}{j} \binom{n-j}{k-j} (-u)^k \\
&= \sum_{k=0}^n \binom{n}{k} (-u)^k \sum_{j=0}^k (-1)^j \binom{k}{j} F(j)
\end{aligned} \tag{11}$$

where the last equality follows from the identity $\binom{n}{j} \binom{n-j}{k-j} = \binom{n}{k} \binom{k}{j}$ and exchanging the order of the sums. The last inner sum may be computed from a known identity for Catalan numbers $C_k = \frac{1}{k+1} \binom{2k}{k}$, namely¹

$$C_k = \sum_{j=0}^k (-1)^j 2^{k-j} \binom{k}{j} \binom{j}{\lfloor j/2 \rfloor} = 2^k \sum_{j=0}^k (-1)^j \binom{k}{j} F(j)$$

which when substituted into (11) yields (10).

By direct verification, the expression on the right of (10) is the hypergeometric function ${}_2F_1(-n, \frac{1}{2}; 2; 2u)$, whose Euler integral representation gives

$$\mathbb{E}[F(S)] = \frac{2}{\pi} \int_0^1 t^{-1/2} (1-t)^{1/2} (1-2ut)^n dt.$$

Multiplying by \sqrt{nu} and using the change of variables $s = 2nut$ we get

$$R^n(p) = \sqrt{nu} \mathbb{E}[F(S)] = \frac{\sqrt{2}}{\pi} \int_0^{2nu} \sqrt{\frac{1}{s} - \frac{1}{2nu}} (1 - \frac{s}{n})^n ds$$

which increases with n towards the limit $\frac{\sqrt{2}}{\pi} \int_0^\infty \frac{1}{\sqrt{s}} e^{-s} ds = \frac{\sqrt{2}}{\pi} \Gamma(\frac{1}{2}) = \sqrt{\frac{2}{\pi}}$. \square

2.3.2 At least one coordinate $p_i = \frac{1}{2}$

With no loss of generality assume $p_1 = \frac{1}{2}$ and denote $S = M_2 + \dots + M_n$. Conditioning on M_1 and setting $g(k) \triangleq \frac{1}{2}[F(k) + F(k+1)]$ we get

$$\mathbb{E}[F(M_1 + \dots + M_n)] = \mathbb{E}[g(S)].$$

A direct calculation shows that $g : \mathbb{N} \rightarrow \mathbb{R}$ is convex, namely

$$g(k) \leq \frac{1}{2}[g(k-1) + g(k+1)] \quad \text{for all } k \geq 1,$$

¹See <http://mathworld.wolfram.com/CatalanNumber.html>. A proof is also given in §4.2.

so we may use the Hoeffding-type inequality in Proposition 12 to obtain $\mathbb{E}[g(S)] \leq \mathbb{E}[g(Z)]$ with $Z \sim P(z)$ a Poisson variable with $z = p_2 + \cdots + p_n$. From this it follows that

$$\begin{aligned} R^n(p) &\leq \sqrt{z + \frac{1}{2}} \mathbb{E}[g(Z)] \\ &= \frac{1}{2} \sqrt{z + \frac{1}{2}} \sum_{k=0}^{\infty} [F(k) + F(1+k)] \exp(-z) \frac{z^k}{k!} \\ &= \sqrt{z + \frac{1}{2}} \exp(-z) [I_0(z) + (1 - \frac{1}{2z}) I_1(z)] \end{aligned} \quad (12)$$

where $I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} (\frac{z}{2})^{2k}$ and $I_1(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} (\frac{z}{2})^{2k+1}$ are modified Bessel functions.

Proposition 9. *Let $h(z)$ denote the expression in (12). Then $h(z)$ is increasing with $h(z) \leq \lim_{z \rightarrow \infty} h(z) = \sqrt{\frac{2}{\pi}}$.*

Proof. The identities $I'_0(z) = I_1(z)$ and $I'_1(z) = I_0(z) - \frac{1}{z} I_1(z)$ imply

$$h'(z) = \frac{\exp(-z)}{4z^2 \sqrt{z + \frac{1}{2}}} [2(1+z)I_1(z) - zI_0(z)]$$

so that proving that h is increasing reduces to $zI_0(z) \leq 2(1+z)I_1(z)$. Letting $x = z/2$ and rearranging terms, this is equivalent to

$$\sum_{k=1}^{\infty} \frac{x^{2k+1}}{(k-1)!(k+1)!} \leq 2 \sum_{k=0}^{\infty} \frac{x^{2k+2}}{k!(k+1)!}.$$

This latter inequality follows easily by noting that each term on the left can be bounded from above by two consecutive terms on the right, namely

$$\frac{x^{2k+1}}{(k-1)!(k+1)!} \leq \frac{x^{2k}}{(k-1)!k!} + \frac{x^{2k+2}}{k!(k+1)!}$$

which results from the trivial inequality $kx \leq k(k+1) + x^2$.

Thus $h(z)$ is increasing and therefore it is bounded from above by its limit $\ell = \lim_{z \rightarrow \infty} h(z)$. To prove that $\ell = \sqrt{\frac{2}{\pi}}$ one may use the known asymptotics $\exp(-z) \sqrt{z} I_\alpha(z) \rightarrow \frac{1}{\sqrt{2\pi}}$ (see [1, Chapter 9]). Alternatively, one may use the integral representation $I_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta) e^{z \cos \theta} d\theta$ to write

$$\ell = \lim_{z \rightarrow \infty} \frac{1}{\pi} \sqrt{z + \frac{1}{2}} \int_0^\pi [1 + (1 - \frac{1}{2z}) \cos \theta] e^{-z(1 - \cos \theta)} d\theta.$$

Since $\frac{1}{2z}\sqrt{z+\frac{1}{2}} \rightarrow 0$ the relevant term for the limit is $\int_0^\pi [1 + \cos \theta] e^{-z(1-\cos \theta)} d\theta$, which is transformed by the change of variables $z(1-\cos \theta) = x^2/2$ into

$$\ell = \lim_{z \rightarrow \infty} \frac{2}{\pi} \sqrt{1 + \frac{1}{2z}} \int_0^{\sqrt{4z}} \left(1 - \frac{x^2}{4z}\right)^{1/2} e^{-x^2/2} dx = \frac{2}{\pi} \int_0^\infty e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}}. \quad \square$$

REMARK. An alternative proof of the monotonicity of $h(z)$ is obtained by substituting the well-known recurrence $I_{n+1} = I_{n-1} - \frac{2n}{z} I_n$ into the Turan-type inequality $I_{n-1} I_{n+1} \leq I_n^2$ (see [29]) which gives $I_{n-1}^2 - \frac{2n}{z} I_{n-1} I_n \leq I_n^2$. Denoting $x = I_{n-1}/I_n$ we have $x^2 - \frac{2n}{z} x \leq 1$, and solving the quadratic we get $x \leq \frac{n}{z} + \sqrt{1 + (\frac{n}{z})^2}$. For $n = 1$ this last expression is smaller than $2(z+1)/z$ which gives $zI_0(z) \leq 2(z+1)I_1(z)$ so that $h'(z) \geq 0$.

2.4 Conclusion

The bounds in §2.3 establish (6) and prove Theorem 1. Moreover, the bound (6) is sharp and cannot be improved. Indeed, for $\alpha_i \equiv \alpha$ constant, setting $u = 2\alpha(1-\alpha)$ and $S \sim B(n, u)$ we have

$$\sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)} P^n = \sqrt{\frac{nu}{2}} \mathbb{E}[F(S)]$$

and by Proposition 8 this quantity converges to $\frac{1}{\sqrt{\pi}}$ as $n \rightarrow \infty$. This does not mean that (3) is itself sharp since we only have $\|x_n - Tx_n\| \leq P^n$. Thus, a natural question is to find the smallest constant κ for which (2) holds. Although we do not know whether (3) is sharp or not, the following example shows that this bound cannot be improved by more than 17%.

EXAMPLE. Take $X = \ell^1(\mathbb{N})$ and let C be the set of all sequences $x = (x^i)_{i \in \mathbb{N}}$ with $x^i \geq 0$ and $\sum_{i=0}^\infty x^i \leq 1$, so that $\text{diam}(C) = 2$. Let $T : C \rightarrow C$ be the right-shift isometry $T(x^0, x^1, x^2, \dots) = (0, x^0, x^1, x^2, \dots)$. Then, the iteration (KM) started from $x_0 = (1, 0, 0, \dots)$ generates a sequence of the form $x_n = (p_n^0, p_n^1, \dots, p_n^n, 0, 0, \dots)$ with

$$p_n^i = \mathbb{P}(X_1 + \dots + X_n = i)$$

where X_i are independent Bernoullis with $\mathbb{P}(X_i = 1) = \alpha_i$. It follows that

$$\begin{aligned} \|x^n - Tx^n\|_1 &= p_n^0 + |p_n^1 - p_n^0| + |p_n^2 - p_n^1| + \dots + |p_n^n - p_n^{n-1}| + p_n^n \\ &= 2 \max\{p_n^i : 0 \leq i \leq n\}. \end{aligned}$$

Now, consider $n = 2m$ Bernoullis trials, half of them with success probability $\alpha_i = \frac{u}{m}$ and the other half with $\alpha_i = 1 - \frac{u}{m}$. Then

$$\max\{p_n^i : 0 \leq i \leq n\} \geq p_{2m}^m = \mathbb{P}(X = Y)$$

with X, Y independent Binomials $B(m, \frac{u}{m})$. When $m \rightarrow \infty$ these Binomials converge to Poissons so that p_{2m}^m tends to $\sum_{k=0}^{\infty} (\frac{\exp(-u)u^k}{k!})^2 = \exp(-2u)I_0(2u)$. Since $\sqrt{\sum_{i=1}^{2m} \alpha_i(1-\alpha_i)}$ tends to $\sqrt{2u}$, it follows that $p_{2m}^m \sqrt{\sum_{i=1}^{2m} \alpha_i(1-\alpha_i)}$ can be made as close as desired to the value $\eta = \max_{x \geq 0} \sqrt{x} \exp(-x) I_0(x)$. Hence the optimal κ lies in the interval $[\eta, \frac{1}{\sqrt{\pi}}] \sim [0.4688, 0.5642]$ which leaves a margin of at most 17%.

3 Two direct applications of Theorem 1

3.1 Convergence of the iterates

The following result, which is basically known (*cf.* [7, 14, 15, 17, 18, 25]), shows how Theorem 1 can be used to obtain the convergence of the iterates, proving at the same time the existence of fixed points.

Proposition 10. *Suppose $\sum \alpha_k(1-\alpha_k) = \infty$ and x_k bounded.*

- (a) *If x_k is relatively compact then $x_k \rightarrow \bar{x}$ for some $\bar{x} \in \text{Fix}(T)$.*
- (b) *If X is a Hilbert space then $x_k \rightharpoonup \bar{x}$ for some $\bar{x} \in \text{Fix}(T)$.*

Proof. (a) Choose a convergent subsequence $x_{k_n} \rightarrow \bar{x}$. From (3) we obtain $x_k - Tx_k \rightarrow 0$ so that \bar{x} must be a fixed point. Since

$$\|x_k - \bar{x}\| = \|(1-\alpha_k)(x_{k-1} - \bar{x}) + \alpha_k(Tx_{k-1} - T\bar{x})\| \leq \|x_{k-1} - \bar{x}\|$$

we conclude that $\|x_k - \bar{x}\|$ decreases to 0.

(b) Since $I - T$ is maximal monotone and $x_k - Tx_k \rightarrow 0$, all weak cluster points of x_k belong to $\text{Fix}(T)$. As before $\|x_k - \bar{x}\|$ converges for all $\bar{x} \in \text{Fix}(T)$ so that weak convergence follows from Opial's lemma. \square

3.2 Unbounded domains

When C is unbounded (2) says nothing. However, if $\text{Fix}(T) \neq \emptyset$ is nonempty², then for each $y \in \text{Fix}(T)$ we may still apply (2) on the bounded subset $\tilde{C} = C \cap B(y, \|y - x_0\|)$ which satisfies $T(\tilde{C}) \subseteq \tilde{C}$ and $\text{diam}(\tilde{C}) \leq 2\|y - x_0\|$. Hence, setting $\tilde{\kappa} = 2\kappa$ and taking the infimum over $y \in \text{Fix}(T)$ we obtain

$$\|x_n - Tx_n\| \leq \tilde{\kappa} \frac{\text{dist}(x_0, \text{Fix}(T))}{\sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)}}. \quad (13)$$

In particular, Theorem 1 implies that (13) holds with $\tilde{\kappa} = \frac{2}{\sqrt{\pi}} \sim 1.1284$. In Hilbert spaces, [30, Vaisman] established a sharper bound with $\tilde{\kappa} = 1$. We present this result which exploits the well-known identity

$$\|(1-\alpha)u + \alpha v\|^2 = (1-\alpha)\|u\|^2 + \alpha\|v\|^2 - \alpha(1-\alpha)\|u - v\|^2. \quad (14)$$

Proposition 11. *Let $T : C \rightarrow C$ be non-expansive on a convex $C \subset E$ with E a Hilbert space and $\text{Fix}(T)$ nonempty. Then (13) holds with $\tilde{\kappa} = 1$.*

Proof. It is known that $\|x_k - Tx_k\|$ decreases with k . Indeed,

$$\begin{aligned} \|x_k - Tx_k\| &= \|(1-\alpha_k)x_{k-1} + \alpha_kTx_{k-1} - Tx_k\| \\ &\leq (1-\alpha_k)\|x_{k-1} - Tx_{k-1}\| + \|Tx_{k-1} - Tx_k\| \\ &\leq (1-\alpha_k)\|x_{k-1} - Tx_{k-1}\| + \|x_{k-1} - x_k\| \\ &= (1-\alpha_k)\|x_{k-1} - Tx_{k-1}\| + \alpha_k\|x_{k-1} - Tx_{k-1}\| \\ &= \|x_{k-1} - Tx_{k-1}\|. \end{aligned}$$

Now, using (14), for each $y \in \text{Fix}(T)$ we get

$$\begin{aligned} \|x_i - y\|^2 &= \|(1-\alpha_i)(x_{i-1} - y) + \alpha_i(Tx_{i-1} - Ty)\|^2 \\ &= (1-\alpha_i)\|x_{i-1} - y\|^2 + \alpha_i\|Tx_{i-1} - Ty\|^2 - \alpha_i(1-\alpha_i)\|x_{i-1} - Tx_{i-1}\|^2 \\ &\leq \|x_{i-1} - y\|^2 - \alpha_i(1-\alpha_i)\|x_{i-1} - Tx_{i-1}\|^2. \end{aligned}$$

Summing these inequalities we see that

$$\sum_{i=1}^n \alpha_i(1-\alpha_i)\|x_{i-1} - Tx_{i-1}\|^2 \leq \|x_0 - y\|^2 - \|x_n - y\|^2$$

²A necessary and sufficient condition to have $\text{Fix}(T) \neq \emptyset$ is that the iterate sequence $\{x_k\}$ remains bounded (cf. [24]).

and the monotonicity of $\|x_k - Tx_k\|$ yields

$$\|x_n - Tx_n\| \sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)} \leq \|x_0 - y\|.$$

The conclusion follows by taking the infimum over $y \in \text{Fix}(T)$. \square

REMARK: The previous proof yields a slightly sharper estimate

$$\|x_{n-1} - Tx_{n-1}\| \leq \frac{\text{dist}(x_0, \text{Fix}(T))}{\sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)}}$$

with x_{n-1} in place of x_n on the left.

4 Auxiliary results

4.1 A Hoeffding-type inequality

In this short section we establish a Hoeffding-type inequality for sums of Bernoullis and Poisson variables. We consider an integer function $g : \mathbb{N} \rightarrow \mathbb{R}$ satisfying the convexity inequalities $g(k) \leq \frac{1}{2}[g(k-1) + g(k+1)]$ for all $k \geq 1$.

Proposition 12. *Let $S = X_1 + \dots + X_m$ be a sum of independent Bernoulli trials with success probabilities $\mathbb{P}(X_i = 1) = p_i$, and let $z = \mathbb{E}(S) = p_1 + \dots + p_m$. Then $\mathbb{E}[g(S)] \leq \mathbb{E}[g(Z)]$ where $Z \sim P(z)$ is a Poisson with the same mean.*

Proof. Let us first note that the expected value $\mathbb{E}[g(S)]$ increases if we replace any variable X_i by a sum $X'_i + X''_i$ of independent Bernoullis with

$$\mathbb{P}(X'_i = 1) = \mathbb{P}(X''_i = 1) = \frac{p_i}{2}.$$

Indeed, for $k \in \mathbb{N}$ let $A(k) = \mathbb{E}[g(k + X_i)]$ and $B(k) = \mathbb{E}[g(k + X'_i + X''_i)]$ so that

$$\begin{aligned} A(k) &= (1 - p_i)g(k) + p_i g(k+1) \\ B(k) &= (1 - \frac{p_i}{2})^2 g(k) + p_i(1 - \frac{p_i}{2})g(k+1) + (\frac{p_i}{2})^2 g(k+2). \end{aligned}$$

Taking their difference we have

$$B(k) - A(k) = (\frac{p_i}{2})^2 [g(k) - 2g(k+1) + g(k+2)] \geq 0$$

so that replacing k by the random variable $\sum_{j \neq i} X_j$ and taking expectation we obtain the asserted monotonicity.

Now, a well-known result by Hoeffding [16, Theorem 3] proves that³ $\mathbb{E}[g(S)] \leq \mathbb{E}[g(S_1)]$ with $S_1 \sim B(n, p)$ a binomial with $p = \frac{1}{n}(p_1 + \dots + p_n)$. Writing S_1 as a sum of n Bernoullis $B(p)$ and sequentially replacing each term by two Bernoullis $B(\frac{p}{2})$, the expected value increases in each step and we get $\mathbb{E}[g(S)] \leq \mathbb{E}[g(S_2)]$ with $S_2 \sim B(2n, p/2)$. Iterating this doubling argument we obtain $\mathbb{E}[g(S)] \leq \mathbb{E}[g(S_k)]$ where $S_k \sim B(2^k n, p/2^k)$. Since $\mathbb{E}(S_k) = z$ for all k , the result follows by letting $k \rightarrow \infty$ and noting that S_k converges to a Poisson variable $Z \sim P(z)$. \square

4.2 An identity for Catalan numbers

In proving Proposition 8 we used the identity

$$C_k = \sum_{j=0}^k (-1)^j 2^{k-j} \binom{k}{j} \binom{j}{\lfloor j/2 \rfloor}.$$

Since this is not found in standard textbooks, for completeness we provide a proof. For each $a \in \mathbb{Z}$ and $P(x)$ a Laurent polynomial (*i.e.* a function whose Laurent series has finitely many terms) we denote by $[x^a]P(x)$ the coefficient of x^a in $P(x)$. We observe that for each non-negative integer j we have

$$\begin{aligned} [x^0](x^2 + x^{-2})^j &= \begin{cases} \binom{j}{\frac{j}{2}} & \text{for } j \text{ even} \\ 0 & \text{for } j \text{ odd} \end{cases} \\ [x^2](x^2 + x^{-2})^j &= \begin{cases} 0 & \text{for } j \text{ even} \\ \binom{j}{\frac{j-1}{2}} & \text{for } j \text{ odd} \end{cases} \end{aligned}$$

so we can write $\binom{j}{\lfloor j/2 \rfloor} = ([x^0] + [x^2])(x^2 + x^{-2})^j$ and therefore

$$\begin{aligned} \sum_{j=0}^k (-1)^j 2^{k-j} \binom{k}{j} \binom{j}{\lfloor j/2 \rfloor} &= ([x^0] + [x^2]) \sum_{j=0}^k \binom{k}{j} 2^{k-j} (-x^2 - x^{-2})^j \\ &= ([x^0] + [x^2]) (2 - x^2 - x^{-2})^k \\ &= ([x^0] + [x^2]) (-(x^1 - x^{-1})^2)^k \\ &= ([x^0] + [x^2]) (-1)^k (x^1 - x^{-1})^{2k} \\ &= \binom{2k}{k} - \binom{2k}{k+1} = C_k. \end{aligned}$$

³As a matter of fact, Hoeffding assumes g strictly convex but the general case follows by applying his result to $g(x) + \epsilon x^2$ with $\epsilon \downarrow 0$.

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